Advanced ODE-Lecture 13
LaSalle's Invariance Principle

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Motivation

- LaSalle Invariance Principle is not only important in limit sets and attractors, but also a foundation stone of modern Lyapunov stability theory.
- LaSalle Invariance Principle is an extension of Krasovskii's Theorem.

LaSalle's Invariance Principle

Still consider a dynamic system

$$x' = f(x)\,. \qquad (13.1)$$

Theorem 13.1 (LaSalle's Invariance Principle) Let $K \subset D$ be a compact, positively invariant set and $V : D \to R$ be of C^1 such that $V'(x) \le 0$ in K.

Let $S = \{x \in K \mid V'(x) = 0\}$ and M be the largest invariant set in S. Then every solution $x(t; x_0)$ of (13.1) starting in K approaches M as $t \to \infty$.

Proof. Let $x(t; x_0)$ be a solution of (13.1), $x_0 \in K$ and $\Omega^+(x_0)$ be a positively limit set.

Step 1. To show that $\lim_{t\to\infty} V(x(t; x_0))$ exists. Since $V'(x) \le 0$ in K, $V(x(t; x_0))$ is decreasing in t. Since $V(x)$ is continuous on a compact set K, it is bounded from below on K. Therefore, $\lim_{t\to\infty} V(x(t; x_0)) = a$ exists.

Step 2. If we show that $\Omega^+(x_0)$ is invariant in S $\Rightarrow$ $\Omega^+(x_0) \subset M$ since M is the largest invariant set in S, and then,

$$\Rightarrow \ \Omega^+(x_0) \subset M \subset S \subset K .$$

Since $\Omega^+(x_0)$ is invariant in K, so we only need to show that $\Omega^+(x_0) \subset S$.

We only need to show that $x(t;x_0) \in S$. For $V(x)$ on $\Omega^+(x_0)$, we have

$$\lim_{t\to\infty} V(x(t;x_0)) = a \quad \Rightarrow \quad V'(x(t;x_0)) \equiv 0 .$$

That is $x(t;x_0) \in S$ $\Rightarrow$ $\Omega^+(x_0) \subset S$. Therefore, $\Omega^+(x_0)$ is invariant in S.

Since $\Omega^+(x_0) \subset M$, $x(t;x_0) \to \Omega^+(x_0)$ as $t \to \infty$ by Lemma 12.2, Hence, $x(t;x_0) \to M$ as $t \to \infty$. □

Remark 13.1

1) If $M = \{0\}$, then $x(t; x_0) \to 0$ as $t \to \infty$ $\Rightarrow$ $x = 0$ is AS if it is stable;

However, the structure of the maximum invariant set M of S may be complex in general.

2) The key of the proof is to use the invariance of $\Omega^+(x_0)$ to locate its position

$$\Omega^+(x_0) \subset M \subset S \subset K;$$

3) In practice, it is difficult to construct K, which is compact and positively invariant, so we often use the bounded $\Omega_c = \{x \mid V(x) \le c\}$ to replace K while to find $V(x) > 0$ and $V'(x) \le 0$ although to find K does not have to be tied in with $V(x)$ in general by LaSalle's invariance principle; If $V(x)$ is not positive definite, Ω_c is not necessarily bounded, for example, $\Omega_c = \{x \mid (x_1 - x_2)^2 \le c\}$ is not bounded no matter how small $c > 0$ is. If $V(x)$ is radially unbounded, the set Ω_c is bounded for all $c > 0$, even for $V(x)$ is not positive definite.

4) $V(x)$ does not have to be positively definite in LaSalle's invariance principle.

5) $x(t;x_0)$ approaches M as $t \to \infty$ doesn't imply that $\lim_{t\to\infty} x(t;x_0)$ exists! For example, the stable limit cycle is the positive limit set of every trajectory starting sufficiently near the limit cycle. However, the solution does not approach any specific point on the limit cycle.

6) Not only does LaSalle's invariance principle relax the negative definiteness requirement of classical Lyapunov theorem, but also it extends Lyapunov theorem in three different directions. First, it gives an estimate of the region of attraction, which is not necessarily of the form $\Omega_c = \{x \mid V(x) \le c\}$. K in LaSalle's invariant principle can be any compact positively invariant set. Second, LaSalle's invariance principle can be used in cases where the system has an equilibrium set, rather than an isolated equilibrium. Third, $V(x) > 0$ is not necessarily true.

An Adaptive Control Example

Example 13.1 Consider the first order equation

$$y' = a y + u$$

together with the (adaptive) control law

$$u = k y, \quad k = \gamma y^2, \quad \gamma > 0 .$$

Taking $x_1 = y$ and $x_2 = k$, the closed-loop system is represented by

$$\begin{cases} x_1' = -(x_2 - a)x_1 \\ x_2' = \gamma x_1^2 \end{cases} .$$

The line $x_1 = 0$ is an equilibrium set (line). We hope to show that the trajectories approach this equilibrium set as $t \to \infty$ for which Krasovskii's theorem and classical Lyapunov theory are not applicable. That means that the adaptive controller drives (regulates) y to zero.

Consider a Lyapunov function candidate as follows.

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2,$$

where $b > a$. The above candidate is not positive definite! But it is suitable for LaSalle invariance principle. The derivative of $V(x)$ along trajectories of the closed-loop system is given by

$$V'(x) = x_1 x_1' + \frac{1}{\gamma}(x_2 - b)x_2' = -x_1^2(b - a) \le 0.$$

Since $V(x)$ is radially unbounded, $\Omega_c = \{x \in R^2 \mid V(x) \le c\}$ for any $c > 0$ is a compact (bounded and closed), positively invariant set. Then, take $K = \Omega_c$. All conditions of LaSalle's invariance principle are satisfied.

$$S = \{x \in \Omega_c \mid x_1 = 0\}$$

is an invariant set because any point on $x_1 = 0$ is an equilibrium point. Therefore, $M = S$. By LaSalle's invariance principle, any trajectory starting in Ω_c approaches S as $t \to \infty$.

Moreover, Since $V(x)$ is radially unbounded, the conclusion is global because for any $x(0) \in R^2$, we can choose $c > 0$ large enough that $x(0) \in \Omega_c \subseteq B_r \subset R^2$ and r exists.

Remark 13.2 In adaptive control, the constant a is not known in general. The condition $b > a$ is not implementable, but we know the existence of b. However, if we know that $|a| \leq \alpha$, where the bound α is known, we can choose that $b > \alpha$.

Corollaries

Corollary 13.2 Let $V : D \to R$ be C^1, such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\};$$

$$V'(x) \le 0 \text{ in } D.$$

Let $S = \{x \in R^n \mid V'(x) = 0\}$. If there is no trajectory that can stay identically in S, other than the origin. Then, the origin of (13.1) is AS.

Proof. We only need to show that the origin is attractive. Taking $K = \Omega_c = \{x \in D \mid V(x) \le c\}$ in LaSalle's invariance principle, which is compact positively invariant since $V(x) > 0$. Then, $V'(x) \le 0$ in $\Omega_c \subset D$, then, any solution $x(t; x_0)$ starting in Ω_c approaches M as $t \to \infty$ by LaSalle's invariance principle. Since $\Omega^+(x_0) \subset M \subset S \subset \Omega_c$, $x(t; x_0) \to \Omega^+(x_0)$ as $t \to \infty$, while $\Omega^+(x_0) = \{0\}$ by Krasovskii theorem. So the origin is attractive.

Corollary 13.3 Let $V: D \to R$ be C^1, such that

$$V(0)=0 \text{ and } V(x)>0 \text{ in } D-\{0\};$$

$$V'(x) \leq 0 \text{ in } D;$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty.$$

Let $S=\{x \in R^n \mid V'(x)=0\}$. If there is no trajectory that can stay identically in S, other than the origin. Then, the origin of (13.1) is GAS.

Proof. For any $x(0) \in R^n$, let $c=V(x(0))>0$. The radially unbounded condition implies that there is $r>0$ such that $\Omega_c \subset B_r$. Then, take $K=\Omega_c$, the rest of proof is the same to Corollary 13.2.

Remark 13.3 Corollary 13.2 and Corollary 13.3 are Kraosovskii theorem for local and global respectively.

Summary

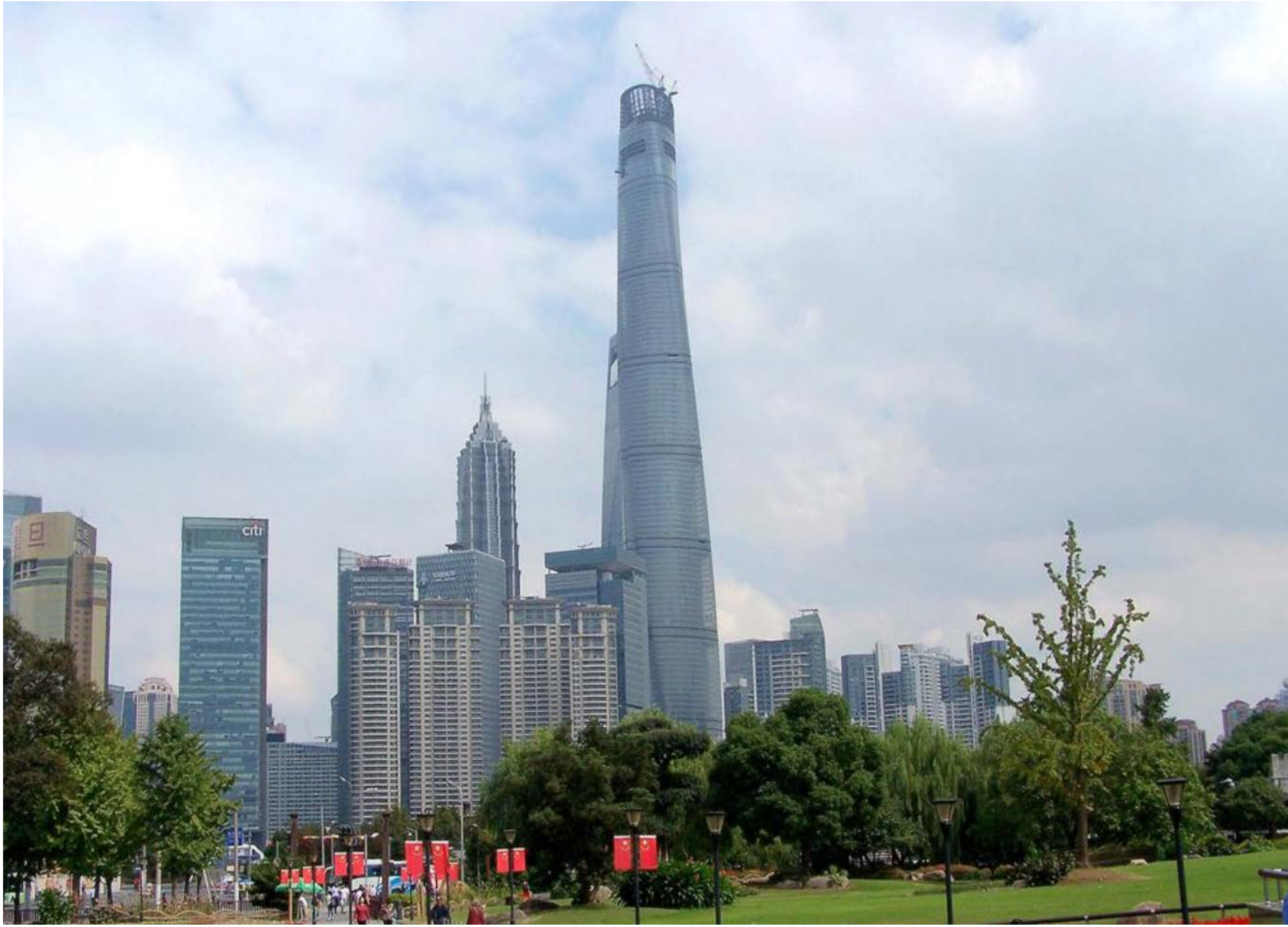
1) LaSalle's Invariance Principle is a symbol of modern Lyapunov stability theory and has many applications in control.

2) The application scope of LaSalle's Invariance Principle is much larger than the one of Krasovskii theorem. However, how to find a compact positively invariant set K is difficult in general.

3) The proofs of both LaSalle's Invariance Principle and Krasovskii theorem are closely depend on the properties of dynamic systems. So it is not trivial to extend these results to time-varying systems.

Homework

Reviews today’s class.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